QUANTUM FAMILIES OF QUANTUM GROUP HOMOMORPHISMS

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ABSTRACT. The notion of a quantum family of maps has been introduced in the framework of C*-algebras. As in the classical case, one may consider a quantum family of maps preserving additional structures (e.g. quantum family of maps preserving a state). In this paper we define a quantum family of homomorphisms of locally compact quantum groups. Roughly speaking, we show that such a family is classical. The purely algebraic counterpart of the discussed notion, i.e. a quantum family of homomorphisms of Hopf algebras, is introduced and the algebraic counterpart of the aforementioned result is proved. Moreover, we show that a quantum family of homomorphisms of Hopf algebras is consistent with the counits and coinverses of the given Hopf algebras. We compare our concept with *weak coactions* introduced by Andruskiewitsch and we apply it to the analysis of adjoint coaction.

1. Introduction

Gelfand-Najmark theorem identifies a commutative C*-algebra A with the algebra of continuous, vanishing at infinity functions on the spectrum $\mathrm{Sp}(A)$. This identification is a source of many concepts in the theory of non-commutative C*-algebras, e.g. quantum family of maps between quantum spaces. The notion was introduced in [14] and further developed in [11], [13]. For the recent survey article on this subject we refer to [9]. Since we also consider the purely algebraic counterpart of the concept, we prefer to use the name *quantum family of morphisms* instead of quantum family of maps between quantum spaces.

Given C*-algebras A_1, A_2 and B, a quantum family of morphisms from A_1 to A_2 is a morphism $\alpha \in \mathrm{Mor}(A_1, B \otimes A_2)$. For $B = \mathrm{C}_0(X)$, α yields a continuous family of morphisms indexed by the elements of X. Indeed, denoting the character assigned to $x \in X$ by $\mathrm{ev}_x : \mathrm{C}_0(X) \to \mathbb{C}$, we may view α as a continuous map

$$X \ni x \mapsto \alpha_x \in \mathrm{Mor}(A_1, A_2)$$

where

$$\alpha_x = (\mathrm{ev}_x \otimes \mathrm{id}_{A_2}) \circ \alpha.$$

For the description of topology on $\mathrm{Mor}(A_1, A_2)$ we refer to [15].

Given quantum family of morphisms may be consistent with certain structures on A_1 and A_2. For instance if $A_1 = A_2 = A$ then $\alpha \in \mathrm{M}(A, B \otimes A)$ is said to preserve a state $\omega : A \to \mathbb{C}$ if

$$(\mathrm{id} \otimes \omega) \circ \alpha(a) = \omega(a)\mathbb{1}_B.$$

In this paper we consider the case when A_1 and A_2 are C*-algebras assigned to locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$ and we define a quantum family of homomorphisms from $\mathbb{H}$ to $\mathbb{G}$ as quantum family of morphisms satisfying further conditions.

In order to explain our concept, let us consider a classical family of homomorphisms from $\mathbb{H}$ to $\mathbb{G}$, i.e. $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0(X) \otimes \mathrm{C}_0^u(\mathbb{H}))$ such that

$$(\alpha_x \otimes \alpha_x) \circ \Delta^u_{\mathbb{G}} = \Delta^u_{\mathbb{H}} \circ \alpha_x \tag{1.1}$$

where $\mathrm{C}_0^u(\mathbb{H}), \mathrm{C}_0^u(\mathbb{G})$ denotes the universal C*-algebras assigned to $\mathbb{H}, \mathbb{G}$ respectively. Let $\mathbb{W}^G \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$ be the universal bicharacter of $\mathbb{G}$ and let us consider a unitary

$$U = (\mathrm{id} \otimes \alpha)(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0(X) \otimes \mathrm{C}_0^u(\mathbb{H})). \tag{1.2}$$

2010 *Mathematics Subject Classification.* Primary 16T30.
Key words and phrases. Quantum families, homomorphisms.
This work was supported by the Harmonia NCN grant 2012/06/M/ST1/00169.

Then, using the *leg numbering notation*, (1.1) is equivalent with the following identity for U

$$(\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_{\mathrm{C}_0(X)} \otimes \Delta^u_{\mathbb{H}})(U) = U_{123}U_{124}. \tag{1.3}$$

This inspires us to define a quantum family of homomorphisms of quantum groups, as quantum family of morphisms

$$\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), B \otimes \mathrm{C}_0^u(\mathbb{H}))$$

such that, defining $U = (\mathrm{id} \otimes \alpha)(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes B \otimes \mathrm{C}_0^u(\mathbb{H}))$ we have

$$(\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_B \otimes \Delta_{\mathrm{C}_0^u(\mathbb{H})})(U) = U_{123}U_{124}$$

where we used the the leg numbering notation. We prove that a quantum family α of homomorphisms of quantum groups satisfies

$$\alpha(x)_{12}\alpha(y)_{13} = \alpha(y)_{13}\alpha(x)_{12}$$

for all $x, y \in \mathrm{C}_0^u(\mathbb{G})$ (see Theorem 2.3). A special case $\mathbb{H} = \mathbb{G}$ of this result was already obtained in [5] and it led to a proof that a quantum group of automorphisms of a finite quantum group as defined in [2] is its classical group of automorphisms.

In the second part of the paper we move on to a purely algebraic context of the category *Alg* of unital algebras over a field k. Let $\mathcal{H}_1 = (\mathcal{A}_1, \Delta_{\mathcal{A}_1}, S_{\mathcal{A}_1}, \varepsilon_{\mathcal{A}_1})$, $\mathcal{H}_2 = (\mathcal{A}_2, \Delta_{\mathcal{A}_2}, S_{\mathcal{A}_2}, \varepsilon_{\mathcal{A}_2})$ be Hopf algebras and $\mathcal{B}$ an algebra. The multiplication map on $\mathcal{B}$ will be denoted $m_{\mathcal{B}} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$. In order to formulate the definition of a quantum family $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ of homomorphisms, let us consider an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of Hopf algebras homomorphisms $\alpha_i : \mathcal{A}_1 \to \mathcal{A}_2$ where

$$\Delta_{\mathcal{A}_2} \circ \alpha_i = (\alpha_i \otimes \alpha_i) \circ \Delta_{\mathcal{A}_1}. \tag{1.4}$$

Denoting the commutative algebra k^n by $\mathcal{B}$ and identifying $\mathcal{B} \otimes \mathcal{A}_2 \cong \mathcal{A}_2^n$ we may view the n-tuple $(\alpha_1(a), \alpha_2(a), \ldots, \alpha_n(a))$ as $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$

$$\alpha(a) = (\alpha_1(a), \alpha_2(a), \ldots, \alpha_n(a)).$$

The condition (1.4) reads

$$(m_{\mathcal{B}} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_{\mathcal{A}_2 \otimes \mathcal{B}} \otimes \mathrm{id}) \circ (\alpha \otimes \alpha) \circ \Delta_{\mathcal{A}_1} = (\mathrm{id} \otimes \Delta_{\mathcal{A}_2}) \circ \alpha \tag{1.5}$$

where

$$\sigma_{\mathcal{A}_2, \mathcal{B}} : \mathcal{A}_2 \otimes \mathcal{B} \to \mathcal{B} \otimes \mathcal{A}_2$$

is the flip homomorphism

$$\sigma_{\mathcal{A}_2, \mathcal{B}}(a \otimes b) = b \otimes a.$$

Allowing $\mathcal{B}$ to be an arbitrary algebra over k, we define a quantum family of Hopf algebra homomorphisms to be a unital homomorphism $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ satisfying (1.5). Let us note that for $\mathcal{B} = k^n$ we have

$$\alpha(x)_{12}\alpha(y)_{13} = \alpha(y)_{13}\alpha(x)_{12} \tag{1.6}$$

for all $x, y \in \mathcal{A}_1$ and

$$\begin{aligned} (\mathrm{id} \otimes S_{\mathcal{A}_2})(\alpha(a)) &= \alpha(S_{\mathcal{A}_1}(a)) \\ (\mathrm{id} \otimes \varepsilon_{\mathcal{A}_2})(\alpha(a)) &= \varepsilon_{\mathcal{A}_1}(a)\mathbb{1}_{\mathcal{B}} \end{aligned} \tag{1.7}$$

for all $a \in \mathcal{A}_1$. Indeed, (1.6) follows from the commutativity of k^n while (1.7) is equivalent with

$$\begin{aligned} S_{\mathcal{A}_2}(\alpha_i(a)) &= \alpha_i(S_{\mathcal{A}_1}(a)) \\ \varepsilon_{\mathcal{A}_2}(\alpha_i(a)) &= \varepsilon_{\mathcal{A}_1}(a) \end{aligned}$$

for all $i \in \{1, 2, \ldots, n\}$. We prove that (1.6) and (1.7) hold for α satisfying (1.5) with arbitrary $\mathcal{B}$ (see Theorem 3.4, Theorem 3.5 and Theorem 3.6). Thus we answer positively the question if (1.7) follows from (1.5) which appeared in [2].

The concept of quantum family of homomorphisms is closely related with Andruskiewitsch's *weak coaction* (see [1]) which is explained in Remark 1. Since the latter was used for the discussion of adjoint coaction our results has some consequences in this context (see Section 3.1). As a side considerations we introduce the concept of cocentralizer of a given algebra homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ and we characterize it in terms of the adjoint coaction.

The results of this paper have certain similarities with the results obtained in [4]. The latter concerns a Hopf algebra $\mathcal{Q}$ coacting on a unital commutative algebra $\mathcal{A}$. The authors formulate some conditions (phrased in terms of a bilinear form which is preserved under the coaction) which forces the Hopf algebra $\mathcal{Q}$ to be commutative. Viewing the coaction as a quantum family of maps of $\mathcal{A}$ the analogy of this result with the context of our paper is clear.

2. A quantum family of homomorphisms of locally compact quantum groups

Let $\mathcal{C}^*$ be the category of C*-algebras. The objects of $\mathcal{C}^*$ are C*-algebras and a morphism $\pi \in \mathrm{Mor}(A, B)$ is a non-degenerate $*$-homomorphism: $\pi : A \to \mathrm{M}(B)$, $\pi(A)B = B$. The tensor product $A \otimes B$ is the spatial one. For the discussion of morphisms and tensor product in the context of $\mathcal{C}^*$-category we refer to [15].

Definition 2.1. Let $A_1, A_2, B \in \mathrm{Obj}(\mathcal{C}^*)$. A morphism $\alpha \in \mathrm{Mor}(A_1, B \otimes A_2)$ will be called a *quantum family of morphisms* from A_1 to A_2.

In what follows we shall introduce and use certain elements of the theory of locally compact quantum groups (lcqg). For the axiomatic formulations of lcqg with the existence of Haar weights postulated we refer to [6] or [7]. For the theory with the multiplicative unitary playing the central role we refer to [10]. For the needs of this paper the theory based on multiplicative unitary is sufficient. We shall freely use the leg numbering notation.

Let $\mathbb{G}$ be a locally compact quantum group and $\hat{\mathbb{G}}$ its dual. The multiplicative unitary $\mathrm{W}^{\mathbb{G}}$ of $\mathbb{G}$ may be viewed as (the reduced) bicharacter $\mathrm{W}^{\mathbb{G}} \in \mathrm{M}(\mathrm{C}_0(\hat{\mathbb{G}}) \otimes \mathrm{C}_0(\mathbb{G}))$.

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}})(\mathrm{W}^{\mathbb{G}}) = \mathrm{W}^{\mathbb{G}}_{12}\mathrm{W}^{\mathbb{G}}_{13}$$
$$(\Delta_{\hat{\mathbb{G}}} \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}) = \mathrm{W}^{\mathbb{G}}_{23}\mathrm{W}^{\mathbb{G}}_{13}.$$

Let B be a C*-algebra and $\pi_1, \pi_2 \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{G}), B)$. If

$$(\mathrm{id} \otimes \pi_1)(\mathrm{W}^{\mathbb{G}}) = (\mathrm{id} \otimes \pi_2)(\mathrm{W}^{\mathbb{G}})$$

then $\pi_1 = \pi_2$ which easily follows from

$$\mathrm{C}_0(\mathbb{G}) = \{(\omega \otimes \mathrm{id})(\mathrm{W}^{\mathbb{G}}) : \omega \in \mathrm{B}(L^2(\mathbb{G}))_*\}^{-\|\cdot\|}.$$

One assigns with $\mathbb{G}$ and $\hat{\mathbb{G}}$ their universal versions. For the constructions of the universal version within the multiplicative unitary framework see [10]. Thus one constructs $\mathrm{C}^u_0(\mathbb{G}), \mathrm{C}^u_0(\hat{\mathbb{G}})$, the reducing morphisms $\Lambda_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}^u_0(\mathbb{G}), \mathrm{C}_0(\mathbb{G}))$, $\Lambda_{\hat{\mathbb{G}}} \in \mathrm{Mor}(\mathrm{C}^u_0(\hat{\mathbb{G}}), \mathrm{C}_0(\hat{\mathbb{G}}))$ and comultiplications

$$\Delta^u_{\mathbb{G}} \in \mathrm{Mor}(\mathrm{C}^u_0(\mathbb{G}), \mathrm{C}^u_0(\mathbb{G}) \otimes \mathrm{C}^u_0(\mathbb{G}))$$
$$\Delta^u_{\hat{\mathbb{G}}} \in \mathrm{Mor}(\mathrm{C}^u_0(\hat{\mathbb{G}}), \mathrm{C}^u_0(\hat{\mathbb{G}}) \otimes \mathrm{C}^u_0(\hat{\mathbb{G}})).$$

It turns out that $\mathrm{W}^{\mathbb{G}}$ lifts to the universal bicharacter $\mathbb{W}^{\mathbb{G}} \in \mathrm{M}(\mathrm{C}^u_0(\hat{\mathbb{G}}) \otimes \mathrm{C}^u_0(\mathbb{G}))$ (see e.g. [8]), i.e.

$$\begin{aligned}(\mathrm{id} \otimes \Delta^u_{\mathbb{G}})(\mathbb{W}^{\mathbb{G}}) &= \mathbb{W}^{\mathbb{G}}_{12}\mathbb{W}^{\mathbb{G}}_{13}\\ (\Delta^u_{\hat{\mathbb{G}}} \otimes \mathrm{id})(\mathbb{W}^{\mathbb{G}}) &= \mathbb{W}^{\mathbb{G}}_{23}\mathbb{W}^{\mathbb{G}}_{13}\end{aligned} \tag{2.1}$$

and

$$\mathrm{W}^{\mathbb{G}} = (\Lambda_{\hat{\mathbb{G}}} \otimes \Lambda_{\mathbb{G}})(\mathbb{W}^{\mathbb{G}}).$$

As in the reduced case, if $\pi_1, \pi_2 \in \mathrm{Mor}(\mathrm{C}^u_0(\mathbb{G}), B)$ satisfies

$$(\mathrm{id} \otimes \pi_1)(\mathbb{W}^{\mathbb{G}}) = (\mathrm{id} \otimes \pi_2)(\mathbb{W}^{\mathbb{G}}) \tag{2.2}$$

then $\pi_1 = \pi_2$ which is the consequence of the equality

$$\mathrm{C}^u_0(\mathbb{G}) = \{(\omega \circ \Lambda_{\hat{\mathbb{G}}} \otimes \mathrm{id}_{\mathrm{C}^u_0(\mathbb{G})})(\mathbb{W}^{\mathbb{G}}) : \omega \in \mathrm{B}(L^2(\mathbb{G}))_*\}^{-\|\cdot\|}. \tag{2.3}$$

Let $\mathbb{H}$ and $\mathbb{G}$ be locally compact quantum groups and let $\alpha \in \mathrm{Mor}(\mathrm{C}^u_0(\mathbb{G}), \mathrm{C}^u_0(\mathbb{H}))$. We say that α is a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ if

$$(\alpha \otimes \alpha) \circ \Delta^u_{\mathbb{G}} = \Delta^u_{\mathbb{H}} \circ \alpha.$$

A morphism $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\mathbb{H}))$ may be assigned with a unitary

$$U = (\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \alpha)(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{H})).$$

Clearly, α is a homomorphism from $\mathbb{H}$ to $\mathbb{G}$ if and only if

$$(\mathrm{id} \otimes \Delta_{\mathbb{H}}^u)(U) = U_{12}U_{13}.$$

Definition 2.2. Let B be a C*-algebra and $\mathbb{H}$, $\mathbb{G}$ locally compact quantum groups. Let $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), B \otimes \mathrm{C}_0^u(\mathbb{H}))$ and let us define $U = (\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \alpha)(\mathbb{W}^{\mathbb{G}}) \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes B \otimes \mathrm{C}_0^u(\mathbb{H}))$. We say that α is a *quantum family of homomorphisms* from $\mathbb{H}$ to $\mathbb{G}$ if U satisfies

$$(\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}}} \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u)(U) = U_{123}U_{124}. \tag{2.4}$$

Let us note that the second equation of (2.1) yields

$$(\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})})(U) = U_{234}U_{134}. \tag{2.5}$$

Theorem 2.3. *Let $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), B \otimes \mathrm{C}_0^u(\mathbb{H}))$ be a quantum family of homomorphisms from $\mathbb{H}$ to $\mathbb{G}$. Then*

$$\alpha_{12}(x)\alpha_{13}(y) = \alpha_{13}(y)\alpha_{12}(x)$$

for all $x, y \in \mathrm{C}_0^u(\mathbb{G})$.

Proof. The morphism

$$\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u \in \mathrm{Mor}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes B \otimes \mathrm{C}_0^u(\mathbb{H}), \mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes B \otimes \mathrm{C}_0^u(\mathbb{H}) \otimes \mathrm{C}_0^u(\mathbb{H}))$$

may be written in two ways

$$\begin{aligned} \Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u &= (\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})} \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})}) \circ (\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u) \\ \Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id} \otimes \Delta_{\mathbb{H}}^u &= (\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u) \circ (\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})}). \end{aligned}$$

Applying these two forms of $\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u$ to $U \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes B \otimes \mathrm{C}_0^u(\mathbb{H}))$ and using (2.4) and (2.5) we get

$$(\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u)(U) = U_{234}U_{134}U_{235}U_{135}$$

and

$$(\Delta_{\hat{\mathbb{G}}}^u \otimes \mathrm{id}_B \otimes \Delta_{\mathbb{H}}^u)(U) = U_{234}U_{235}U_{134}U_{135}.$$

In particular

$$U_{134}U_{235} = U_{235}U_{134} \tag{2.6}$$

Let $\mu, \nu \in \mathrm{C}_0^u(\hat{\mathbb{G}})^*$. Applying $\mu \otimes \nu \otimes \mathrm{id}_B \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})} \otimes \mathrm{id}_{\mathrm{C}_0^u(\mathbb{H})}$ to (2.6) and using (2.3) we get

$$\alpha(x)_{12}\alpha(y)_{13} = \alpha(y)_{13}\alpha(x)_{12}$$

for all $x, y \in \mathrm{C}_0^u(\mathbb{G})$. ◻

2.1. **Adjoint coaction.** Let $\mathbb{G}$ be a locally compact quantum group and let us denote the universal bicharacter of $\mathbb{G}$ by $\mathbb{W}^{\mathbb{G}} \in \mathrm{M}(\mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$. Let us consider a quantum family of maps $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$ given by

$$\alpha(x) = \mathbb{W}^{\mathbb{G}}(\mathbb{1}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes x)\mathbb{W}^{\mathbb{G}*}$$

for all $x \in \mathrm{C}_0^u(\mathbb{G})$.

Theorem 2.4. *A quantum family of maps $\alpha \in \mathrm{Mor}(\mathrm{C}_0^u(\mathbb{G}), \mathrm{C}_0^u(\hat{\mathbb{G}}) \otimes \mathrm{C}_0^u(\mathbb{G}))$ is a quantum family of homomorphisms from $\mathbb{G}$ to $\mathbb{G}$ if and only if for all $x \in \mathrm{C}_0^u(\mathbb{G})$ and $y \in \mathrm{C}_0^u(\hat{\mathbb{G}})$ we have*

$$\alpha(x)(y \otimes \mathbb{1}) = (y \otimes \mathbb{1})\alpha(x).$$

Proof. Noting that $U = (\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \alpha)(\mathbb{W}^{\mathbb{G}}) = \mathbb{W}^{\mathbb{G}}_{23}\mathbb{W}^{\mathbb{G}}_{13}\mathbb{W}^{\mathbb{G}*}_{23}$ we compute

$$\begin{aligned}(\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \Delta^u_{\mathbb{G}})(U) &= \mathbb{W}^{\mathbb{G}}_{23}\mathbb{W}^{\mathbb{G}}_{24}\mathbb{W}^{\mathbb{G}}_{13}\mathbb{W}^{\mathbb{G}}_{14}\mathbb{W}^{\mathbb{G}*}_{24}\mathbb{W}^{\mathbb{G}*}_{23} \\ &= \mathbb{W}^{\mathbb{G}}_{23}\mathbb{W}^{\mathbb{G}}_{13}\mathbb{W}^{\mathbb{G}}_{24}\mathbb{W}^{\mathbb{G}}_{14}\mathbb{W}^{\mathbb{G}*}_{24}\mathbb{W}^{\mathbb{G}*}_{23}.\end{aligned}$$

On the other hand

$$U_{123}U_{124} = \mathbb{W}^{\mathbb{G}}_{23}\mathbb{W}^{\mathbb{G}}_{13}\mathbb{W}^{\mathbb{G}*}_{23}\mathbb{W}^{\mathbb{G}}_{24}\mathbb{W}^{\mathbb{G}}_{14}\mathbb{W}^{\mathbb{G}*}_{24}.$$

The equality $(\mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \mathrm{id}_{\mathrm{C}_0^u(\hat{\mathbb{G}})} \otimes \Delta^u_{\mathbb{G}})(U) = U_{123}U_{124}$ yields

$$\mathbb{W}^{\mathbb{G}}_{24}\mathbb{W}^{\mathbb{G}}_{14}\mathbb{W}^{\mathbb{G}*}_{24}\mathbb{W}^{\mathbb{G}*}_{23} = \mathbb{W}^{\mathbb{G}*}_{23}\mathbb{W}^{\mathbb{G}}_{24}\mathbb{W}^{\mathbb{G}}_{14}\mathbb{W}^{\mathbb{G}*}_{24}$$

which we may write as

$$\mathbb{W}^{\mathbb{G}}_{23}(\mathrm{id} \otimes \alpha)(\mathbb{W}^{\mathbb{G}})_{124} = (\mathrm{id} \otimes \alpha)(\mathbb{W}^{\mathbb{G}})_{124}\mathbb{W}^{\mathbb{G}}_{23}.$$

Reasoning as at the end of the proof of Theorem 2.3, we get the desired result. □

Let us note that

- if $\mathrm{C}_0^u(\mathbb{G})$ is commutative then the action α is trivial and the condition of Theorem 2.4 holds;
- if $\mathrm{C}_0^u(\mathbb{G})$ is cocommutative (thus $\mathrm{C}_0^u(\hat{\mathbb{G}})$ is commutative) then the condition of Theorem 2.4 holds.

3. Hopf algebraic context

Let k be a field and *Alg* the category of unital algebras over k. The tensor product of $\mathcal{B},\mathcal{C} \in \mathrm{Obj}(\mathit{Alg})$ is denoted $\mathcal{B} \otimes \mathcal{C}$ and the flip morphism by $\sigma_{\mathcal{B},\mathcal{C}} : \mathcal{B} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{B}$. The multiplication map of $\mathcal{B}$ is denoted $m_{\mathcal{B}} : \mathcal{B} \otimes \mathcal{B} \to \mathcal{B}$; $m_{\mathcal{B}}$ is a morphism if and only if $\mathcal{B}$ is commutative.

A Hopf algebra $\mathcal{H}$ is a quadruple $\mathcal{H} = (\mathcal{A}, \Delta_{\mathcal{A}}, \varepsilon_{\mathcal{A}}, S_{\mathcal{A}})$ satisfying the standard system of axioms (see [12]). In what follows we shall use Sweedler notation

$$\Delta_{\mathcal{A}_1}(a) = a_{(1)} \otimes a_{(2)}$$

when convenient. Using this notation we have for instance

$$\begin{aligned} a_{(1)}S_{\mathcal{A}}(a_{(2)}) &= S_{\mathcal{A}}(a_{(1)})a_{(2)} = \varepsilon_{\mathcal{A}}(a)\mathbb{1}_{\mathcal{A}} \\ a_{(1)}\varepsilon_{\mathcal{A}}(a_{(2)}) &= \varepsilon_{\mathcal{A}}(a_{(1)})a_{(2)} = a \end{aligned} \tag{3.1}$$

for all $a \in \mathcal{A}$.

Given a k-vector space $\mathcal{V}$, $\mathcal{L}(\mathcal{V})$ denotes the algebra of endomorphism of $\mathcal{V}$. For the later needs we formulate the following simple lemma.

Lemma 3.1. *Let $\mathcal{H}$ be a Hopf algebra, $\mathcal{V}$ a vector space and $T_1, T_2 : \mathcal{A} \to \mathcal{V}$ the linear maps such that*

$$(\mathrm{id}_{\mathcal{A}} \otimes T_1) \circ \Delta_{\mathcal{A}} = (\mathrm{id}_{\mathcal{A}} \otimes T_2) \circ \Delta_{\mathcal{A}}.$$

Then $T_1 = T_2$. Similarly,

$$(T_1 \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta_{\mathcal{A}} = (T_2 \otimes \mathrm{id}_{\mathcal{A}}) \circ \Delta_{\mathcal{A}}$$

implies $T_1 = T_2$.

Proof. For all $a \in \mathcal{A}$ we have

$$T_1(a) = (\varepsilon_{\mathcal{A}} \otimes T_1) \circ \Delta_{\mathcal{A}}(a) = (\varepsilon_{\mathcal{A}} \otimes T_2) \circ \Delta_{\mathcal{A}}(a) = T_2(a)$$

thus $T_1 = T_2$. □

The motivation of the following definition was given in the Introduction.

Definition 3.2. Let $\mathcal{B}$ be an algebra, $\mathcal{H}_1 = (\mathcal{A}_1, \Delta_{\mathcal{A}_1}, S_{\mathcal{A}_1}, \varepsilon_{\mathcal{A}_1})$, $\mathcal{H}_2 = (\mathcal{A}_2, \Delta_{\mathcal{A}_2}, , S_{\mathcal{A}_2}, \varepsilon_{\mathcal{A}_2})$ Hopf algebras and let $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ be a morphism. We say that α is a *quantum family of homomorphisms* from $\mathcal{H}_1$ to $\mathcal{H}_2$ if

$$(m_{\mathcal{B}} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_{\mathcal{A}_2 \otimes \mathcal{B}} \otimes \mathrm{id}) \circ (\alpha \otimes \alpha) \circ \Delta_{\mathcal{A}_1} = (\mathrm{id} \otimes \Delta_{\mathcal{A}_2}) \circ \alpha. \tag{3.2}$$

Let us note that in the Sweedler notation, the condition of Definition 3.2 reads

$$\alpha(a_{(1)})_{12}\alpha(a_{(2)})_{13} = (\mathrm{id}\otimes\Delta_{\mathcal{A}_2})\circ\alpha(a) \tag{3.3}$$

for all $a\in\mathcal{A}_1$.

Remark 1. A similar concept was considered in the beginning of Paragraph 2.3 of [1] in the following context. Let $(\mathcal{A},\Delta_\mathcal{A},\varepsilon_\mathcal{A})$ be a coalgebra and $(\mathcal{B},\Delta_\mathcal{B},S_\mathcal{B},\varepsilon_\mathcal{B})$ a Hopf algebra. Let m_{24} : $\mathcal{A}\otimes\mathcal{B}\otimes\mathcal{A}\otimes\mathcal{B}$ be the linear map $c\otimes h\otimes d\otimes k\mapsto c\otimes d\otimes hk$. A linear map $\rho:\mathcal{A}\to\mathcal{A}\otimes\mathcal{B}$ is called a weak coaction if it satisfies the following conditions

$$\begin{aligned}(\Delta_\mathcal{A}\otimes\mathrm{id})\circ\rho &= m_{24}\circ(\rho\otimes\rho)\circ\Delta_\mathcal{A}\\ (\varepsilon_\mathcal{A}\otimes\mathrm{id})\circ\rho &= \varepsilon_\mathcal{A}\otimes\mathbb{1}\\ (\mathrm{id}\otimes\varepsilon_\mathcal{B})\circ\rho &= \mathrm{id}_\mathcal{A}.\end{aligned} \tag{3.4}$$

The first from the above conditions has a form which up to the flip of $\mathcal{B}\otimes\mathcal{A}$ is the same as the condition (3.2) (with $\mathcal{A}_1=\mathcal{A}_2=\mathcal{A}$). Let us note that from the Hopf algebra structure of $\mathcal{B}$ only the counit $\varepsilon_\mathcal{B}$ is used in (3.4). We shall see that the analog of the second condition of (3.4) is automatically satisfied for a quantum family of homomorphisms (see Theorem 3.5).

Let $\mathcal{H}=(\mathcal{A},\Delta_\mathcal{A},S_\mathcal{A},\varepsilon_\mathcal{A})$ be a Hopf algebra. The algebraic counterpart of the multiplicative unitary is the invertible linear map $W^\mathcal{H}:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ such that

$$W^\mathcal{H}(a\otimes a')=\Delta_{\mathcal{A}_1}(a)(\mathbb{1}\otimes a')=a_{(1)}\otimes a_{(2)}a'.$$

The inverse of $W^\mathcal{H}$ is given by

$$(W^\mathcal{H})^{-1}(a\otimes a')=((\mathrm{id}\otimes S_\mathcal{A})\Delta_\mathcal{A}(a))(\mathbb{1}\otimes a')=a_{(1)}\otimes S_\mathcal{A}(a_{(2)})a'.$$

The map $W\in\mathcal{L}(\mathcal{A}\otimes\mathcal{A})$ yields $W_{12},W_{13},W_{23}\in\mathcal{L}(\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A})$, e.g.

$$W_{12}(a\otimes b\otimes c)=a_{(1)}\otimes a_{(2)}b\otimes c.$$

Let us note that

$$W^\mathcal{H}_{23}W^\mathcal{H}_{12}(W^\mathcal{H}_{23})^{-1}=W_{12}W_{13}.$$

Indeed, using the Sweedler notation we get

$$W^\mathcal{H}_{23}W^\mathcal{H}_{12}(a\otimes b\otimes c)=(a_{(1)}\otimes a_{(2)}\otimes a_{(3)})(\mathbb{1}\otimes b_{(1)}\otimes b_{(2)}c)=W^\mathcal{H}_{12}W^\mathcal{H}_{13}W^\mathcal{H}_{23}(a\otimes b\otimes c)$$

for all $a,b,c\in\mathcal{A}$.

Theorem 3.3. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hopf algebras, $\mathcal{B}$ a unital algebra and $\alpha:\mathcal{A}_1\to\mathcal{B}\otimes\mathcal{A}_2$ a morphism. Let us consider a linear map $U:\mathcal{A}_1\otimes\mathcal{B}\otimes\mathcal{A}_2\to\mathcal{A}_1\otimes\mathcal{B}\otimes\mathcal{A}_2$ such that*

$$U(a\otimes b\otimes a')=(\mathrm{id}\otimes\alpha)(\Delta_{\mathcal{A}_1}(a))(\mathbb{1}\otimes b\otimes a')$$

for all $a\in\mathcal{A}_1, b\in\mathcal{B}, a'\in\mathcal{A}_2$. Then U is invertible,

$$U^{-1}(a\otimes b\otimes a')=(\mathrm{id}\otimes\alpha)((\mathrm{id}\otimes S_{\mathcal{A}_1})\Delta_{\mathcal{A}_1}(a))(\mathbb{1}\otimes b\otimes a')$$

and it satisfies

$$(W^{\mathcal{H}_1}_{12})^{-1}U_{234}W^{\mathcal{H}_1}_{12}=U_{134}U_{234}. \tag{3.5}$$

Moreover, α is a quantum family of homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$ if and only if

$$W^{\mathcal{H}_2}_{34}U_{123}(W^{\mathcal{H}_2}_{34})^{-1}=U_{123}U_{124}. \tag{3.6}$$

Proof. Let $X:\mathcal{A}_1\otimes\mathcal{B}\otimes\mathcal{A}_2\to\mathcal{A}_1\otimes\mathcal{B}\otimes\mathcal{A}_2$ be a linear map such that

$$X(a\otimes b\otimes c)=(\mathrm{id}\otimes\alpha)((\mathrm{id}\otimes S_{\mathcal{A}_1})\Delta_{\mathcal{A}_1}(a))(\mathbb{1}\otimes b\otimes c) \tag{3.7}$$

for all $a \in \mathcal{A}_1, b \in \mathcal{B}, c \in \mathcal{A}_2$. Using the Sweedler notation, we compute

$$\begin{aligned} UX(a \otimes b \otimes c) &= U((a_{(1)} \otimes \alpha(S_{\mathcal{A}_1}(a_{(2)})))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\ &= (a_{(1)} \otimes \alpha(a_{(2)})\alpha(S_{\mathcal{A}_1}(a_{(3)})))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\ &= (a_{(1)} \otimes \alpha(a_{(2)} S_{\mathcal{A}_1}(a_{(3)})))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\ &= (a_{(1)} \otimes \alpha(\varepsilon(a_{(2)})\mathbb{1}_{\mathcal{A}_1}))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\ &= (a_{(1)}\varepsilon(a_{(2)}) \otimes b \otimes c \\ &= a \otimes b \otimes c \end{aligned}$$

where in the third and the fifth equation we used (3.1). This shows that $UX = \mathrm{id}_{\mathcal{A}_1 \otimes \mathcal{B} \otimes \mathcal{A}_2}$. Similarly we check that $XU = \mathrm{id}_{\mathcal{A}_1 \otimes \mathcal{B} \otimes \mathcal{A}_2}$.

For $a, b \in \mathcal{A}_1$, $c \in \mathcal{B}$ and $d \in \mathcal{A}_2$ we have

$$\begin{aligned} U_{234} W_{12}^{\mathcal{H}_1}(a \otimes b \otimes c \otimes d) &= U_{234}(a_{(1)} \otimes a_{(2)} b \otimes c \otimes d) \\ &= (a_{(1)} \otimes a_{(2)} \otimes \alpha(a_{(3)}))(\mathbb{1} \otimes b_{(1)} \otimes \alpha(b_{(2)}))(\mathbb{1} \otimes \mathbb{1} \otimes c \otimes d). \end{aligned}$$

On the other hand

$$\begin{aligned} W_{12}^{\mathcal{H}_1} U_{134} U_{234}(a \otimes b \otimes c \otimes d) &= W_{12}^{\mathcal{H}_1} U_{134}(a \otimes b_{(1)} \otimes \alpha(b_{(2)}))(\mathbb{1} \otimes \mathbb{1} \otimes c \otimes d) \\ &= W_{12}^{\mathcal{H}_1}(a_{(1)} \otimes \mathbb{1} \otimes \alpha(a_{(2)}))(\mathbb{1} \otimes b_{(1)} \otimes \alpha(b_{(2)}))(\mathbb{1} \otimes \mathbb{1} \otimes c \otimes d) \\ &= (a_{(1)} \otimes a_{(2)} \otimes \alpha(a_{(3)}))(\mathbb{1} \otimes b_{(1)} \otimes \alpha(b_{(2)}))(\mathbb{1} \otimes \mathbb{1} \otimes c \otimes d) \end{aligned}$$

which proves (3.5).

Let $a \in \mathcal{A}_1$, $b \in \mathcal{B}$ and $c, d \in \mathcal{A}_2$. We compute

$$\begin{aligned} W_{34}^{\mathcal{H}_2} U_{123} (W_{34}^{\mathcal{H}_2})^{-1}(a \otimes b \otimes c \otimes d) &= W_{34}^{\mathcal{H}_2} U_{123}(a \otimes b \otimes c_{(1)} \otimes S_{\mathcal{A}_2}(c_{(2)})d) \\ &= (a_{(1)} \otimes ((\mathrm{id} \otimes \Delta_{\mathcal{A}_2})(\alpha(a_{(2)}))))(\mathbb{1} \otimes b \otimes c \otimes d). \end{aligned}$$

On the other hand we get

$$\begin{aligned} U_{123} U_{124}(a \otimes b \otimes c \otimes d) &= U_{123}(a_{(1)} \otimes \alpha(a_{(2)})_{24})(\mathbb{1} \otimes b \otimes c \otimes d) \\ &= (a_{(1)} \otimes \alpha(a_{(2)})_{23} \alpha(a_{(3)})_{24})(\mathbb{1} \otimes b \otimes c \otimes d). \end{aligned}$$

If α is a quantum family of homomorphism, then we may replace the elements $(\mathrm{id} \otimes \Delta_{\mathcal{A}_2})(\alpha(a_{(2)}))$ with $\alpha(a_{(2)})_{23}\alpha(a_{(3)})_{24}$ which yields (3.6).

Conversely, if (3.6) holds then putting $b = \mathbb{1}_{\mathcal{B}}, c = d = \mathbb{1}_{\mathcal{A}_2}$ we get

$$a_{(1)} \otimes \alpha(a_{(2)})_{23}\alpha(a_{(3)})_{24} = a_{(1)} \otimes (\mathrm{id} \otimes \Delta_{\mathcal{A}_2})(\alpha(a_{(2)})).$$

Using Lemma 3.1 we get $\alpha(a_{(1)})_{12}\alpha(a_{(2)})_{13} = (\mathrm{id} \otimes \Delta_{\mathcal{A}_2})(\alpha(a))$ for all $a \in \mathcal{A}_1$, i.e. α is a quantum family of homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$. □

Theorem 3.4. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hopf algebras and $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ a quantum family of homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then for any $a, a' \in \mathcal{A}_1$ we have*

$$\alpha(a)_{12}\alpha(a')_{13} = \alpha(a')_{13}\alpha(a)_{12}.$$

Proof. Using

$$\begin{aligned} W_{34}^{\mathcal{H}_2} U_{123} (W_{34}^{\mathcal{H}_2})^{-1} &= U_{123} U_{124} \\ (W_{12}^{\mathcal{H}_1})^{-1} U_{234} W_{12}^{\mathcal{H}_1} &= U_{134} U_{234} \end{aligned} \tag{3.8}$$

we may see that

$$(W_{12}^{\mathcal{H}_1})^{-1} W_{45}^{\mathcal{H}_2} U_{234} (W_{45}^{\mathcal{H}_2})^{-1} W_{12}^{\mathcal{H}_1} = (W_{12}^{\mathcal{H}_1})^{-1} U_{234} U_{235} W_{12}^{\mathcal{H}_1} = U_{134} U_{234} U_{135} U_{235}.$$

Since $(W_{12}^{\mathcal{H}_1})^{-1} W_{45}^{\mathcal{H}_2} = W_{45}^{\mathcal{H}_2} (W_{12}^{\mathcal{H}_1})^{-1}$ we get the same result computing

$$W_{45}^{\mathcal{H}_2} (W_{12}^{\mathcal{H}_1})^{-1} U_{234} W_{12}^{\mathcal{H}_1} (W_{45}^{\mathcal{H}_2})^{-1} = W_{45}^{\mathcal{H}_2} U_{134} U_{234} (W_{45}^{\mathcal{H}_2})^{-1} = U_{134} U_{135} U_{234} U_{235}.$$

Using the invertibility of U we conclude that

$$U_{234} U_{135} = U_{135} U_{234}. \tag{3.9}$$

Applying (3.9) to a simple tensor $y \otimes x \otimes \mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}_2} \otimes \mathbb{1}_{\mathcal{A}_2}$, with $x, y \in \mathcal{A}_1$ we get

$$(\mathrm{id} \otimes \alpha)(\Delta_{\mathcal{A}_1}(x))_{234}(\mathrm{id} \otimes \alpha)(\Delta_{\mathcal{A}_1}(y))_{135} = (\mathrm{id} \otimes \alpha)(\Delta_{\mathcal{A}_1}(y))_{135}(\mathrm{id} \otimes \alpha)(\Delta_{\mathcal{A}_1}(x))_{234}. \tag{3.10}$$

Applying $(\varepsilon \otimes \varepsilon \otimes \mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id})$ to (3.10) we conclude that

$$\alpha(x)_{12}\alpha(y)_{13} = \alpha(y)_{13}\alpha(x)_{12}$$

for all $x, y \in \mathcal{A}_1$. □

Theorem 3.5. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hopf algebras and $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ a quantum family of homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then*

$$(\mathrm{id} \otimes \varepsilon_{\mathcal{A}_2})(\alpha(a)) = \varepsilon_{\mathcal{A}_1}(a)\mathbb{1}_{\mathcal{B}} \tag{3.11}$$

for all $a \in \mathcal{A}_1$.

Proof. The identity

$$W^{\mathcal{H}_2}_{34} U_{123} (W^{\mathcal{H}_2}_{34})^{-1} = U_{123}U_{124}$$

enables us to prove that

$$(\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}_2})(U(a \otimes b \otimes c)) = \varepsilon_{\mathcal{A}_2}(c) a \otimes b \tag{3.12}$$

for all $a \in \mathcal{A}_1$, $b \in \mathcal{B}$ and $c \in \mathcal{A}_2$. Indeed, let additional $d \in \mathcal{A}_2$. On one hand we have

$$W^{\mathcal{H}_2}_{34} U_{123} (W^{\mathcal{H}_2}_{34})^{-1}(a \otimes b \otimes c \otimes d) = (a_{(1)} \otimes ((\mathrm{id}_{\mathcal{B}} \otimes \Delta_{\mathcal{A}_2})(\alpha(a_{(2)}))))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c \otimes d)$$

thus

$$\begin{aligned}(\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}_2} \otimes \varepsilon_{\mathcal{A}_2})(W^{\mathcal{H}_2}_{34} U_{123} (W^{\mathcal{H}_2}_{34})^{-1}(a \otimes b \otimes c \otimes d)) =\\ = (a_{(1)} \otimes \alpha(a_{(2)}))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c)\varepsilon_{\mathcal{A}_2}(d)\\ = U_{123}(a \otimes b \otimes c)\varepsilon_{\mathcal{A}_2}(d)\end{aligned} \tag{3.13}$$

where in the first equality we used the first equation of (3.1). On the other hand we have

$$\begin{aligned}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \varepsilon_{\mathcal{A}_2})(U_{123}U_{124}(a \otimes b \otimes c \otimes d)) =\\ = U_{123}(\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}_2} \otimes \varepsilon_{\mathcal{A}_2})(U_{124}(a \otimes b \otimes c \otimes d))\end{aligned}$$

which together with (3.13) shows that

$$U_{123}(\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}_2} \otimes \varepsilon_{\mathcal{A}_2})(U_{124}(a \otimes b \otimes c \otimes d)) = U_{123}(a \otimes b \otimes c)\varepsilon_{\mathcal{A}_2}(d). \tag{3.14}$$

Using the invertibility of U we cancel U_{123} in (3.14) and we get (3.12). Putting $b = \mathbb{1}_{\mathcal{B}}, c = \mathbb{1}_{\mathcal{A}_2}$ in (3.12) we get

$$(\mathrm{id}_{\mathcal{A}_1} \otimes ((\mathrm{id}_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}_2}) \circ \alpha)) \circ \Delta_{\mathcal{A}_1}(a)) = a \otimes \mathbb{1}_{\mathcal{B}} = (\mathrm{id}_{\mathcal{A}_1} \otimes \varepsilon_{\mathcal{A}_1}(\cdot)\mathbb{1}_{\mathcal{B}}) \circ \Delta_{\mathcal{A}_1}(a).$$

Using Lemma 3.1 we conclude that

$$(\mathrm{id}_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}_2})(\alpha(a)) = \varepsilon_{\mathcal{A}_1}(a)\mathbb{1}_{\mathcal{B}}$$

for all $a \in \mathcal{A}_1$. □

Theorem 3.6. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hopf algebras and $\alpha : \mathcal{A}_1 \to \mathcal{B} \otimes \mathcal{A}_2$ a quantum family of homomorphisms from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then*

$$\alpha \circ S_{\mathcal{A}_1} = (\mathrm{id}_{\mathcal{B}} \otimes S_{\mathcal{A}_2}) \circ \alpha.$$

Proof. Let us define an operator $Y \in \mathcal{L}(\mathcal{A}_1 \otimes \mathcal{B} \otimes \mathcal{A}_2)$

$$Y(a \otimes b \otimes c) = (a_{(1)} \otimes ((\mathrm{id}_{\mathcal{B}} \otimes S_{\mathcal{A}_2})\alpha(a_{(2)})))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \tag{3.15}$$

where $a \in \mathcal{A}_1, b \in \mathcal{B}, c \in \mathcal{A}_2$. We compute

$$\begin{aligned}
UY(a \otimes b \otimes c) &= U(a_{(1)} \otimes ((\mathrm{id}_{\mathcal{B}} \otimes S_{\mathcal{A}_2})\alpha(a_{(2)})))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) = \\
&= (a_{(1)} \otimes \alpha(a_{(2)}))(\mathbb{1}_{\mathcal{A}_1} \otimes (\mathrm{id}_{\mathcal{B}} \otimes S_{\mathcal{A}_2})\alpha(a_{(3)}))(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\
&= (\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes m_{\mathcal{A}_2}) \left((\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}_2} \otimes S_{\mathcal{A}_2})(a_{(1)} \otimes \alpha(a_{(2)})_{23}\alpha(a_{(3)})_{24})\right) (\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\
&= (\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes m_{\mathcal{A}_2}) \left((\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes \mathrm{id}_{\mathcal{A}_2} \otimes S_{\mathcal{A}_2})(a_{(1)} \otimes (\mathrm{id}_{\mathcal{B}} \otimes \Delta_{\mathcal{A}_2})(\alpha(a_{(2)})))\right) (\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\
&= (a_{(1)} \otimes ((\mathrm{id}_{\mathcal{B}} \otimes \varepsilon_{\mathcal{A}_2})(\alpha(a_{(2)}))) \otimes \mathbb{1}_{\mathcal{A}_2})(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) \\
&= (a_{(1)}\varepsilon_{\mathcal{A}_1}(a_{(2)}) \otimes \mathbb{1}_{\mathcal{B}} \otimes \mathbb{1}_{\mathcal{A}_2})(\mathbb{1}_{\mathcal{A}_1} \otimes b \otimes c) = a \otimes b \otimes c
\end{aligned}$$

where in the fifth equality we used the fact that α is a quantum family of homomorphisms of Hopf algebras; in the sixth equality we used (3.1); in the seventh equality we used (3.11). Since U is invertible, we get $Y = U^{-1}$. Putting $b = \mathbb{1}_{\mathcal{B}}$ and $c = \mathbb{1}_{\mathcal{A}_2}$ in (3.7) and (3.15) we get

$$(\mathrm{id}_{\mathcal{A}_1} \otimes \alpha)((\mathrm{id}_{\mathcal{A}_1} \otimes S_{\mathcal{A}_1})\Delta_{\mathcal{A}_1}(a)) = (\mathrm{id}_{\mathcal{A}_1} \otimes \mathrm{id}_{\mathcal{B}} \otimes S_{\mathcal{A}_2})(\mathrm{id}_{\mathcal{A}_1} \otimes \alpha)(\Delta_{\mathcal{A}_1}(a))$$

for all $a \in \mathcal{A}_1$. Using Lemma 3.1 we get the desired equality. □

3.1. **Adjoint coaction and cocentralizers.** Let $\mathcal{H} = (\mathcal{A}, \Delta_{\mathcal{A}}, \varepsilon_{\mathcal{A}}, S_{\mathcal{A}})$ be a Hopf algebra. The adjoint coaction $ad : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a linear map such that

$$ad(x) = x_{(1)}S_{\mathcal{A}}(x_{(3)}) \otimes x_{(2)}.$$

Note that

$$(\mathrm{id} \otimes ad) \circ ad(x) = x_{(1)}S_{\mathcal{A}}(x_{(5)}) \otimes x_{(2)}S_{\mathcal{A}}(x_{(4)}) \otimes x_{(3)} = (\Delta_{\mathcal{A}} \otimes \mathrm{id}) \circ ad(x).$$

Moreover

$$\begin{aligned}
(m \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_{\mathcal{A}\otimes\mathcal{A}} \otimes \mathrm{id}) \circ (ad \otimes ad) \circ \Delta_{\mathcal{A}}(x) &= x_{(1)}S_{\mathcal{A}}(x_{(3)})x_{(4)}S_{\mathcal{A}}(x_{(6)}) \otimes x_{(2)} \otimes x_{(5)} \\
&= x_{(1)}S_{\mathcal{A}}(x_{(4)}) \otimes x_{(2)} \otimes x_{(3)} \\
&= (\mathrm{id} \otimes \Delta_{\mathcal{A}}) \circ ad(x).
\end{aligned}$$

Thus if ad is an algebra homomorphism then it can be viewed as a quantum family of homomorphisms from $\mathcal{A}$ to $\mathcal{A}$. Using [3, Theorem 4.2] we see that this is the case if and only if $ad(\mathcal{A}) \subset \mathcal{Z}(\mathcal{A}) \otimes \mathcal{A}$ where $\mathcal{Z}(\mathcal{A})$ is the center of $\mathcal{A}$. Thus we get the algebraic counterpart of Theorem 2.4.

In the context of [1] the adjoint coaction was used for the construction of a cocenter of a Hopf algebra (see also [3]). In what follows we shall introduce and discuss the concept of cocentralizer of a given algebra morphism $\Phi : \mathcal{A} \to \mathcal{B}$.

Definition 3.7. Let $\mathcal{B}$ and $\mathcal{C}$ be algebras over a field k and $\Phi : \mathcal{A} \to \mathcal{B}$, $\Psi : \mathcal{A} \to \mathcal{C}$ surjective homomorhisms. We say that Ψ and Φ cocommute if

$$(\Psi \otimes \Phi) \circ \Delta = (\Psi \otimes \Phi) \circ \Delta^{\mathrm{op}}.$$

We say that the surjective algebra homomorophism $\Psi^{\mathrm{u}} : \mathcal{A} \to \mathcal{C}^{\mathrm{u}}$ is the *cocentralizer* of Φ if Ψ^{u} cocommutes with Φ and for each $\Psi : \mathcal{A} \to \mathcal{C}$ cocommuting with Φ there exists a surjective homomorphism $\pi : \mathcal{C}^{\mathrm{u}} \to \mathcal{C}$ such that $\Psi = \pi \circ \Psi^{\mathrm{u}}$.

Proposition 3.8. *Ψ and Φ cocommute if and only if*

$$(\Phi \otimes \Psi)(ad(x)) = \mathbb{1} \otimes \Psi(x) \tag{3.16}$$

for all $x \in \mathcal{A}$.

Proof. Let us consider $\pi_1 : \mathcal{A} \to \mathcal{B} \otimes \mathcal{C}$, $\pi_1(x) = \Phi(x) \otimes \mathbb{1}$ and $\pi_2 : \mathcal{A} \to \mathcal{B} \otimes \mathcal{C}$, $\pi_2(x) = \mathbb{1} \otimes \Psi(x)$. Denoting the convolution of π_1 and π_2 by $\pi_1 * \pi_2$ we see that Φ and Ψ cocommute if and only if $\pi_1 * \pi_2 = \pi_2 * \pi_1$. Let us note that π_1 is convolution invertible and its inverse is $\pi_3(x) = \Phi(S_{\mathcal{A}}(x)) \otimes \mathbb{1}$. Indeed

$$\begin{aligned}
\pi_1 * \pi_3(x) &= \Phi(x_{(1)})\Phi(S_{\mathcal{A}}(x_{(2)})) \otimes \mathbb{1} \\
&= \Phi(x_{(1)})S_{\mathcal{A}}(x_{(2)})) \otimes \mathbb{1} \\
&= \varepsilon(x)(\mathbb{1} \otimes \mathbb{1}).
\end{aligned}$$

Thus Φ and Ψ cocommute if and only if

$$\pi_1 * \pi_2 * \pi_3 = \pi_2$$

which is equivalent with (3.16). □

Remark 2. Let us consider $\alpha : \mathcal{A} \to \mathcal{B} \otimes \mathcal{A}$, $\alpha = (\Phi \otimes \mathrm{id}) \circ ad$. Then α satisfies

$$(m_{\mathcal{B}} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \sigma_{\mathcal{A} \otimes \mathcal{B}} \otimes \mathrm{id}) \circ (\alpha \otimes \alpha) \circ \Delta_{\mathcal{A}} = (\mathrm{id} \otimes \Delta_{\mathcal{A}}) \circ \alpha.$$

Thus flipping $\mathcal{B} \otimes \mathcal{A}$ we can view α as the weak coaction (see also Remark 1). Theorem 3.4 shows that if α is an algebra homomorphism then $\alpha(x)_{12}\alpha(y)_{13} = \alpha(y)_{13}\alpha(x)_{12}$ for all $x, y \in \mathcal{A}$.

In order to give an explicit description of the cocentralizer of Φ let us consider $\Psi : \mathcal{A} \to \mathcal{C}$ cocommuting with Φ. Using Proposition 3.8 we see that

$$\mathcal{I} = \mathrm{linspan}\{(\mu \otimes \mathrm{id})(\alpha(x)) - \mu(\mathbb{1})x : x \in \mathcal{A},\ \mu \in \mathcal{B}^*\} \subset \ker \Psi.$$

Thus defining an ideal $\mathcal{J} = \mathcal{A}\mathcal{I}\mathcal{A} \subset \mathcal{A}$ we get the quotient $\mathcal{A} \to \mathcal{A}/\mathcal{J}$ which can be identified with cocentralizer of Φ in the sens of Definition 3.7. Using results of [1, Section 2.3] we conclude that $\mathcal{A}/\mathcal{J}$ can be equipped with the (unique) bialgebra structure such that the quotient $\Psi^{\mathrm{u}} : \mathcal{A} \to \mathcal{A}/\mathcal{J}$ is the bialgebra map.

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